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XXIV. *On the Structure of the Atom: an Investigation of the Stability and Periods of Oscillation of a number of Corpuscles arranged at equal intervals around the Circumference of a Circle; with Application of the results to the Theory of Atomic Structure. By* J. J. THOMSON, *F.R.S., Cavendish Professor of Experimental Physics, Cambridge* *.

THE view that the atoms of the elements consist of a number of negatively electrified corpuscles enclosed in a sphere of uniform positive electrification, suggests, among other interesting mathematical problems, the one discussed in this paper, that of the motion of a ring of n negatively electrified particles placed inside a uniformly electrified sphere. Suppose when in equilibrium the n corpuscles are arranged at equal angular intervals round the circumference of a circle of radius a, each corpuscle carrying a charge e of negative electricity. Let the charge of positive electricity contained within the sphere be νe, then if b is the radius of this sphere, the radial attraction on a corpuscle due to the positive electrification is equal to $\nu e^2 a/b^3$; if the corpuscles are at rest this attraction must be balanced by the repulsion exerted by the other corpuscles. Now the repulsion along OA, O being the centre of the sphere, exerted on a corpuscle at A by one at B, is equal to $\frac{e^2}{\mathrm{AB}^2}\cos \mathrm{OAB}$, and, if OA = OB, this is equal to $\frac{e^2}{4\mathrm{OA}^2 \sin\frac{1}{2}\mathrm{AOB}}$: hence, if we have n corpuscles arranged at equal angular intervals $2\pi/n$ round the circumference of a circle, the radial repulsion on one corpuscle

* Communicated by the Author.

due to the other $(n-1)$ is equal to

$$\frac{e^2}{4a^2}\left(\operatorname{cosec}\frac{\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}+\operatorname{cosec}\frac{3\pi}{n}+\ldots\quad+\operatorname{cosec}\frac{(n-1)\pi}{n}\right).$$

If the corpuscles are at rest this must be equal to the radial attraction. Hence, if

$$S_n=\operatorname{cosec}\frac{\pi}{n}+\operatorname{cosec}\frac{2\pi}{n}+\ldots\quad\operatorname{cosec}\frac{(n-1)\pi}{n},$$

$$\frac{\nu e^2 a}{b^3}=\frac{e^2}{4a^2}S_n,$$

or

$$\frac{a^3}{b^3}=\frac{S_n}{4\nu}. \qquad (1)$$

The following are the values of S_n from $n=2$ to $n=6$.

$S_2=1$, $S_3=2{\cdot}3094$, $S_4=3{\cdot}8284$, $S_5=5{\cdot}5056$, $S_6=7{\cdot}3094$.

In the important case when $\nu=n$, *i. e.* when the positive charge on the sphere is equal to the sum of all the negative charges in the ring of corpuscles, we get by (1) the following values for a/b :—

n.	$\frac{a}{b}$.
2	·5
3	·5773
4	·6208
5	·6505
6	·6726

If the ring of corpuscles, instead of being at rest, is rotating with an angular velocity ω, the condition for steady motion is

$$\frac{\nu e^2 a}{b^3}=ma\omega^2+\frac{e^2}{4a^2}S_n,$$

or

$$\frac{\nu a^3}{b^3}=\frac{m}{e^2}\omega^2+\frac{S_n}{4};$$

here m is the mass of a corpuscle.

We shall now proceed to find the forces acting on a corpuscle when the corpuscles are slightly displaced from their positions of equilibrium. Let the position of the corpuscles be fixed by the polar coordinates r and θ in the plane of the undisturbed orbit, and by the displacement z at right angles to this plane; let r_s, θ_s, z_s be the coordinates of the sth corpuscle; then, since the corpuscles are but slightly displaced from their positions of equilibrium, $r_s=a+\rho_s$ where ρ_s is small compared with a, z_s is also small compared with a, and $\theta_s-\theta_{s-1}=\frac{2\pi}{n}+\phi_s-\phi_{s-1}$, where n is the number of corpuscles and the ϕ's are small quantities.

The radial repulsion exerted by the *s*th corpuscle on the *p*th is equal to

$$-e^2\frac{d}{dr_p}\frac{1}{(r_p^2+r_s^2-2r_pr_s\cos(\theta_s-\theta_p)+(z_p-z_s)^2)^{\frac{1}{2}}};$$

expanding this, retaining only the first powers of ρ, ϕ, and z, we find that if R_{ps} is this repulsion

$$R_{ps}=\frac{e^2}{4a^2\sin\psi}\left\{1-\frac{\rho_p}{a}\left(\frac{3}{2}-\frac{1}{2\sin^2\psi}\right)-\frac{\rho_s}{a}\left(\frac{1}{2}+\frac{1}{2\sin^2\psi}\right)-\tfrac{1}{2}(\phi_s-\phi_p)\cot\psi\right\},$$

where $\psi=(p-s)\frac{\pi}{n}$.

The tangential force Θ_{ps} tending to increase θ_p is equal to

$$-\frac{e^2}{r_pd\theta_p}\frac{1}{\left\{r_p^2+r_s^2-2r_pr_s\cos(\theta_s-\theta_p)+(z_p-z_s)^2\right\}^{\frac{1}{2}}};$$

expanding this and retaining only the first powers of the small quantities, we get

$$\Theta_{ps}=-\frac{e^2}{4a^2}\frac{\cos\psi}{\sin^2\psi}\left\{1-\frac{3}{2}\frac{\rho_p}{a}-\frac{1}{2}\frac{\rho_s}{a}-(\phi_s-\phi_p)(\cot\psi+\tfrac{1}{2}\tan\psi)\right\}.$$

Z_{ps}, the force at right angles to the undisturbed plane of the orbit, is easily seen to be given by the equation

$$Z_{ps}=\frac{e^2}{8a^3\sin^3\psi}(z_p-z_s).$$

The total radial force R_p exerted on the *p*th corpuscle by all the other corpuscles, is equal to

$$\frac{e^2}{4a^2}S-\rho_\rho A'-\Sigma\rho_{p+s}A_{p\,.\,p+s}-a\Sigma\phi_{p+s}B_{p\,.\,p+s},$$

where
$$S=\frac{1}{\sin\frac{\pi}{n}}+\frac{1}{\sin\frac{2\pi}{n}}+\cdots\frac{1}{\sin\frac{(n-1)\pi}{n}};$$

$$A'=\frac{e^2}{4a^3}\left(\frac{3}{2}\left(\frac{1}{\sin\frac{\pi}{n}}+\frac{1}{\sin\frac{2\pi}{n}}+\cdots\frac{1}{\sin\frac{(n-1)\pi}{n}}\right)-\frac{1}{2}\left(\frac{1}{\sin^3\frac{\pi}{n}}+\frac{1}{\sin^3\frac{2\pi}{n}}+\cdots\frac{1}{\sin^3\frac{(n-1)\pi}{n}}\right)\right);$$

$$A_{p.p+s}=\frac{e^2}{8a^3}\left(\frac{1}{\sin\frac{s\pi}{n}}+\frac{1}{\sin^3\frac{s\pi}{n}}\right);$$

$$B_{p.p+s}=\frac{e^3}{8a^3}\frac{\cos\frac{s\pi}{n}}{\sin^2\frac{s\pi}{n}}.$$

The coefficient of ϕ_p in the expression for R_p vanishes, since

$$\Sigma\frac{\cos\frac{s\pi}{n}}{\sin^2\frac{s\pi}{n}}=0.$$

As $A_{p.p+s}$, $B_{p.p+s}$ do not involve p, it is more convenient to use the symbols A_s and B_s for these quantities, and to write

$$R_p=\frac{e^2}{4a^2}S-\rho_pA'-\Sigma\rho_{p+s}A_s-a\Sigma\phi_{p+s}B_s.$$

The tangential force Θ_p acting on the p'th particle may similarly be written

$$\Theta_p=\Sigma\rho_{p+s}B_s-a\phi_pC+a\Sigma\phi_{p+s}C_s,$$

where

$$C=\frac{e^2}{4a^3}\left(\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\left(\cot\frac{\pi}{n}+\frac{1}{2}\tan\frac{\pi}{n}\right)+\frac{\cos\frac{2\pi}{n}}{\sin^2\frac{2\pi}{n}}\left(\cot\frac{2\pi}{n}+\frac{1}{2}\tan\frac{2\pi}{n}\right)+\ldots\right),$$

$$C_s=\frac{e^2}{4a^3}\frac{\cos\frac{s\pi}{n}}{\sin^2\frac{s\pi}{n}}\left(\cot\frac{s\pi}{n}+\frac{1}{2}\tan\frac{s\pi}{n}\right);$$

while Z_p, the force at right angles to the plane of the orbit, is given by the equation

$$Z_p=z_pD-\Sigma z_{p+s}D_s,$$

where

$$D=\frac{e^2}{8a^3}\left(\frac{1}{\sin^3\frac{\pi}{n}}+\frac{1}{\sin^3\frac{2\pi}{n}}+\cdots\frac{1}{\sin^3\frac{(n-1)\pi}{n}}\right),$$

and

$$D_s=\frac{e^2}{8a^3}\frac{1}{\sin^3\frac{s\pi}{n}}.$$

The equations of motion of the pth corpuscle are

$$m\left(\frac{d^2r_p}{dt^2}-r_p\left(\frac{d\theta_p}{dt}\right)^2\right)=-\frac{ve^2r_p}{b^3}+R_p\,; \quad . \quad . \quad (\alpha)$$

$$m\left(r\frac{d^2\theta_p}{dt^2}+2\frac{dr}{dt}\frac{d\theta_p}{dt}\right)=\Theta_p\,; \quad . \quad . \quad . \quad . \quad . \quad (\beta)$$

$$m\frac{d^2z_p}{dt^2}=-\frac{ve^2}{b^3}z_p+Z_p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\gamma)$$

Retaining only the first powers of small quantities, we get from these equations, if ω is the value of $\frac{d\theta}{dt}$ when the motion is steady,

$$\frac{ve^2a}{b^3}=ma\omega^2+\frac{e^2}{4a^2}S,$$

$$m\frac{d^2\rho_p}{dt^2}-2ma\omega\frac{d\theta_p}{dt}=\rho_p\left(m\omega^2-\frac{ve^2}{b^3}\right)+R_p-\frac{e^2}{4a^2}S.$$

If ρ_p and θ_p vary as $e^{\iota qt}$, this equation may be written

$$(A-mq^2)\rho_p+A_1\rho_{p+1}+A_2\rho_{p+2}+\ldots$$
$$-2ma\omega\iota q\phi_p+aB_1\phi_{p+1}+aB_2\phi_{p+2}+\ldots=0,$$

where

$$A=\frac{e^2}{4a^3}S+A'=\frac{e^2}{8a^3}\left\{5\left(\frac{1}{\sin\frac{\pi}{n}}+\frac{1}{\sin\frac{2\pi}{n}}+\cdots\right)-\left(\frac{1}{\sin^3\frac{\pi}{n}}+\frac{1}{\sin^3\frac{2\pi}{n}}+\cdots\right)\right\}$$

Writing 1, 2, 3 for p we get

$$\left.\begin{array}{l}(A-mq^2)\rho_1+A_1\rho_2+A_2\rho_3\ldots+A_{n-1}\rho_n-2ma\omega\iota q\phi_1+aB_1\phi_2+aB_2\phi_3+\ldots=0\\(A-mq^2)\rho_2+A_1\rho_3+A_2\rho_4+\ldots\ldots\ldots\quad-2ma\omega\iota q\phi_2+aB_1\phi_3+aB_2\phi_4+\ldots=0\\\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\qquad\qquad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\(A-mq^2)\rho_n+A_1\rho_1+A_2\rho_2+\ldots\ldots\ldots\quad-2ma\omega\iota q\phi_n+aB_1\phi_1+aB_2\phi_2+\ldots=0\end{array}\right\}(A)$$

By equation β we have

$$2m\omega\iota q\frac{\rho_p}{a}-B_1\frac{\rho_{p+1}}{a}-B_2\frac{\rho_{p+2}}{a}+\ldots(C-mq^2)\phi_p-C_1\phi_{p+1}-C_2\phi_{p+2}-\ldots=0.$$

Writing 1, 2, 3 in succession for p we get

$$\left.\begin{array}{l}2\iota m\omega q\frac{\rho_1}{a}-B_1\frac{\rho_2}{a}-B_2\frac{\rho_3}{a}\ldots\quad+(C-mq^2)\phi_1-C_1\phi_2-C_2\phi_3-\ldots=0\\2\iota m\omega q\frac{\rho_2}{a}-B_1\frac{\rho_3}{a}-B_2\frac{\rho_4}{a}\ldots\quad+(C-mq^2)\phi_2-C_2\phi_3-C_2\phi_4-\ldots=0\\\cdot\quad\cdot\quad\cdot\quad\cdot\qquad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\2\iota m\omega q\frac{\rho_n}{a}-B_1\frac{\rho_1}{a}-B_2\frac{\rho_2}{a}\ldots\quad+(C-mq^2)\phi_n-C_1\phi_1-C_2\phi_2-\ldots=0\end{array}\right\}(B)$$

To solve equations A and B we notice that if ω be any root of the equation $x^n=1$, *i. e.* if ω be one of the nth roots of unity, equations A will be satisfied by

$\rho_2=\omega\rho_1,\ \rho_3=\omega\rho_2,\ \rho_4=\omega\rho_3\ldots\quad \phi_2=\omega\phi_1,\ \phi_3=\omega\phi_2,\ \phi_4=\omega\phi_3\ldots$
provided

$$\rho_1(\mathrm{A}-mq^2+\omega\mathrm{A}_1+\omega^2\mathrm{A}_2+\ldots\omega^{n-1}\mathrm{A}_{n-1})$$
$$+\phi_1 a(-2\iota m\omega q+\omega\mathrm{B}_1+\omega^2\mathrm{B}_2+\omega^{n-1}\mathrm{B}_{n-1})=0\ ; \quad (1)$$

while equations B will be satisfied by the same values provided

$$\rho_1(2\iota m\omega q-\omega\mathrm{B}_1-\omega^2\mathrm{B}_2-\omega^{n-1}\mathrm{B}_{n-1})$$
$$+\phi_1 a(\mathrm{C}-mq^2-\omega\mathrm{C}_1-\omega^2\mathrm{C}_2-\omega^{n-1}\mathrm{C}_{n-1})=0. \quad (2)$$

Hence, if both sets of equations are satisfied by these values, we have, eliminating ρ_1 and ϕ_1 from (1) and (2),

$$((\mathrm{A}-mq^2)+\omega\mathrm{A}_1+\omega^2\mathrm{A}_2+\ldots\omega^{n-1}\mathrm{A}_{n-1})$$
$$(\mathrm{C}-mq^2-\omega\mathrm{C}_1-\omega^2\mathrm{C}_2-\omega^{n-1}\mathrm{C}_{n-1})$$
$$=-(-2\iota m\omega q+\omega\mathrm{B}_1+\omega^2\mathrm{B}_2+\ldots\omega^{n-1}\mathrm{B}_n)^2, \quad (1)$$

a biquadratic equation to determine q the frequency of the oscillations of the system. Now ω is of the form

$$\cos\frac{2k\pi}{n}+\iota\sin\frac{2k\pi}{n},$$

where k is an integer between 0 and $n-1$. Substituting this value for ω, we find

$$\omega\mathrm{A}_1+\omega^2\mathrm{A}_2+\omega^{n-1}\mathrm{A}_{n-1}=\frac{e^2}{8a^3}\left\{\cos\frac{2k\pi}{n}\left(\frac{1}{\sin\frac{\pi}{n}}+\frac{1}{\sin^3\frac{\pi}{n}}\right)\right.$$
$$\left.+\cos\frac{4k\pi}{n}\left(\frac{1}{\sin\frac{2\pi}{n}}+\frac{1}{\sin^3\frac{2\pi}{n}}\right)+\cos\frac{6k\pi}{n}\left(\frac{1}{\sin\frac{3\pi}{n}}+\frac{1}{\sin^3\frac{3\pi}{n}}\right)+\ldots\right\}$$

We shall denote this by L_k; it will be noticed that L_k contains no imaginary terms. We find also that

$$\omega\mathrm{C}_1+\omega^2\mathrm{C}_2+\omega^3\mathrm{C}_3+\omega^{n-1}\mathrm{C}_{n-1}=\frac{e^2}{4a^3}\left(\cos\frac{2k\pi}{n}\,\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}\left(\cot\frac{\pi}{n}+\frac{1}{2}\tan\frac{\pi}{n}\right)\right.$$
$$+\cos\frac{4k\pi}{n}\,\frac{\cos\frac{2\pi}{n}}{\sin^2\frac{2\pi}{n}}\left(\cot\frac{2\pi}{n}+\frac{1}{2}\tan\frac{2\pi}{n}\right)$$
$$\left.+\ldots\right).$$

We shall denote this by N_k.

Again,

$$\omega B_1+\omega^2 B_2+\omega^{n-1}B_{n-1}=\frac{\iota\,.\,e^2}{8a^3}\left(\sin\frac{2k\pi}{n}\frac{\cos\frac{\pi}{n}}{\sin^2\frac{\pi}{n}}+\sin\frac{4k\pi}{n}\frac{\cos\frac{2\pi}{n}}{\sin^2\frac{2\pi}{n}}+\sin\frac{6k\pi}{n}\frac{\cos\frac{3\pi}{n}}{\sin^2\frac{3\pi}{n}}+\ldots\right)$$

$$=\iota M_k,\text{ say.}$$

Substituting these values, equation (1) becomes

$$((A-mq^2)+L_k)(C-mq^2-N_k)=(M_k-2m\omega q)^2. \quad . \quad (2)$$

From the value of C given on p. 240 we see that C is the value of N_k when $k=0$, and so may be denoted by N_0, and that $A=\frac{3}{4}\frac{e^2}{a^3}S-L_0$; hence equation (2) may be written

$$\left(\frac{3}{4}\frac{e^2}{a^3}S+L_k-L_0-mq^2\right)(N_0-N_k-mq^2)=(M_k-2m\omega q)^2. \quad (3)$$

k in this equation may have any value from 0 to $(n-1)$; but we see that if we write $n-k$ for k, the values of q given by the two equations differ only in sign, and so give the same frequencies; thus all the values of q can be got by putting $k=0, 1,\ldots\frac{n-1}{2}$, if n be odd, or $k=0, 1, \frac{n}{2}$ if n be even; thus if n be odd there are $\frac{n+1}{2}$ equations of the type (3). When $k=0$, $M_k=0$, and (3) reduces to a quadratic equation; so that the number of roots of these $\frac{n+1}{2}$ equations is $4\times\frac{n+1}{2}-2=2n$; if n be even there are $\frac{n}{2}+1$ equations; but as $M_k=0$ when $k=0$ and $k=\frac{n}{2}$, two of these reduce to quadratics; so that the number of roots of these equations is $4\left(\frac{n}{2}+1\right)-4=2n$. Thus in each case the number of roots is equal to $2n$, the number of degrees of freedom of the corpuscles in the plane of their undisturbed orbit.

Let us now consider the motion at right angles to this plane. By equation γ we have

$$m\frac{d^2z_p}{dt^2}=-\frac{\nu e^2}{b^3}z_p+Dz_p-\Sigma D_s z_{p+s};$$

or if z_p is proportional to $\epsilon^{\iota qt}$,

$$z_p\left(\frac{\nu e^2}{b^3}-\mathrm{D}-mq^2\right)+\Sigma \mathrm{D}_s z_{p+s}=0\ ;$$

thus

$$\left.\begin{array}{l} z_1\left(\frac{\nu e^2}{b^3}-\mathrm{D}-mq^2\right)+z_2\mathrm{D}_1+z_3\mathrm{D}_2+\ldots=0, \\ z_2\left(\frac{\nu e^2}{b^3}-\mathrm{D}-mq^2\right)+z_3\mathrm{D}_1+z_4\mathrm{D}_2+\ldots=0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_n\left(\frac{\nu e^2}{b^3}-\mathrm{D}-mq^2\right)+z_1\mathrm{D}_1+z_2\mathrm{D}_2+\ldots=0. \end{array}\right\} \quad . \quad . \quad (\mathrm{C})$$

We see that ω again being one of the nth roots of unity, the solution of equations C is

$$z_2=\omega z_1, \qquad z_3=\omega z_2, \qquad z_4=\omega z_3\ldots$$

$$\frac{\nu e^2}{b^3}-\mathrm{D}-mq^2+\omega\mathrm{D}_1+\omega^2\mathrm{D}_2+\omega^{n-1}\mathrm{D}_{n-1}=0. \quad . \quad . \quad (4)$$

Putting $\omega=\cos\frac{2k\pi}{n}+\iota\sin\frac{2k\pi}{n}$, we find, substituting the values for D given above, that

$$\omega\mathrm{D}_1+\omega^2\mathrm{D}_2+\omega^{n-1}\mathrm{D}_{n-1}=\frac{e^2}{8a^3}\left(\frac{\cos\frac{2k\pi}{n}}{\sin^3\frac{\pi}{n}}+\frac{\cos\frac{4k\pi}{n}}{\sin^3\frac{2\pi}{n}}+\frac{\cos\frac{6k\pi}{n}}{\sin^3\frac{3\pi}{n}}+\ldots\right)$$

Denoting this by P_k and noticing that $\mathrm{D}=\mathrm{P}_0$, we find that equation (4) becomes

$$\frac{\nu e}{b^3}+\mathrm{P}_k-\mathrm{P}_0-mq^2=0.$$

Putting in succession $k=0, 1, \ldots n-1$, we get n values of q giving the n frequencies corresponding to the displacements at right angles to the plane of the undisturbed orbit.

We shall now proceed to calculate the frequencies for systems containing various numbers of corpuscles. The four quantities L_k, M_k, N_k, P_k which occur in the frequency equation may be expressed in terms of three quantities S_k, T_k, U_k, where

$$\mathrm{S}_k=\Sigma_1^{n-1}\cos\frac{2ks\pi}{n}\frac{1}{\sin\frac{s\pi}{n}},$$

$$\mathrm{T}_k = \Sigma_1^{n-1} \cos \frac{2ks\pi}{n} \frac{1}{\sin^3 \frac{s\pi}{n}},$$

$$\mathrm{U}_k = \Sigma_1^{n-1} \sin \frac{2ks\pi}{n} \frac{\cos \frac{s\pi}{n}}{\sin^2 \frac{s\pi}{n}};$$

for we have

$$\mathrm{L}_k = (\mathrm{S}_k + \mathrm{T}_k) \frac{e^2}{8a^3}, \quad \mathrm{N}_k = (2\mathrm{T}_k - \mathrm{S}_k) \frac{e^2}{8a^3},$$

$$\mathrm{M}_k = \mathrm{U}_k \frac{e^2}{8a^3}, \quad \mathrm{P}_k = \mathrm{T}_k \frac{e^2}{8a^3}.$$

Case of two corpuscles.

When $n=2$ we have

$$\mathrm{L}_0 = \frac{2e^2}{8a^3}, \quad \mathrm{M}_0 = 0, \quad \mathrm{N}_0 = \frac{e^2}{8a^3}, \quad \mathrm{P}_0 = \frac{e^2}{8a^3},$$

$$\mathrm{L}_1 = -\frac{2e^2}{8a^3}, \quad \mathrm{M}_1 = 0, \quad \mathrm{N}_1 = -\frac{e^2}{8a^3}, \quad \mathrm{P}_1 = -\frac{e^2}{8a^3}.$$

Hence for vibrations in the plane of the orbit we have, when $k=0$,

$$\left(\frac{3}{4}\frac{e^2}{a^3} - mq^2\right)(-mq^2) = 4m^2\omega^2 q^2;$$

the roots of this equation are

$$q=0, \quad q = \sqrt{\frac{3}{4}\frac{e^2}{ma^3} + 4\omega^2} = \sqrt{\frac{3\nu e^2}{mb^3} + \omega^2}.$$

When $k=1$, the frequency equation is

$$\left(\frac{1}{4}\frac{e^2}{a^3} - mq^2\right)^2 = 4m^2\omega^2 q^2;$$

the roots of this equation are

$$q = \quad \omega \pm \sqrt{\frac{1}{4}\frac{e^2}{ma^3} + \omega^2} = \omega \pm \sqrt{\frac{\nu e^2}{mb^3}}$$

and

$$q = -\omega \pm \sqrt{\frac{1}{4}\frac{e^2}{ma^3} + \omega^2} = -\omega \pm \sqrt{\frac{\nu e^2}{mb^3}},$$

the second set of values only differing in sign from the first.

For the vibrations perpendicular to the plane of the orbit, we have for $k=0$,

$$q=\sqrt{\frac{\nu e^2}{mb^3}};$$

for $k=1$,

$$q=\sqrt{\frac{\nu e^2}{mb^3}-\frac{e^2}{4ma^3}}=\omega.$$

Thus the six frequencies corresponding to the six degrees of freedom of the two corpuscles are

$$0,\ \omega,\ \sqrt{\frac{\nu e}{mb^3}},\ \sqrt{\frac{\nu e^2}{mb^3}}-\omega,\ \sqrt{\frac{\nu e^2}{mb^3}}+\omega,\ \sqrt{\frac{3\nu e^2}{mb^3}+\omega^2}.$$

When the corpuscles are not rotating round the circle, two of these roots are zero, three equal to $\sqrt{\frac{\nu e^2}{mb^3}}$, and the sixth equal to $\sqrt{\frac{3\nu e^2}{mb^3}}$. Thus the effect of rotation on the triple frequency $\sqrt{\nu e^2/mb^3}$ is to separate the roots, one remaining unaltered, one increasing, and the other diminishing.

Case of three corpuscles.

When $n=3$.

$$S_0=\frac{4}{\sqrt{3}},\quad T_0=\frac{16}{3\sqrt{3}},\quad U_0=0,\quad L_0=\frac{28}{3\sqrt{3}}\frac{e^2}{8a^3},$$

$$N_0=\frac{20}{3\sqrt{3}}\frac{e^2}{8a^3},\quad M_0=0,\quad P_0=\frac{16}{3\sqrt{3}}\frac{e^2}{8a^3},$$

$$S_1=-\frac{2}{\sqrt{3}},\quad T_1=-\frac{8}{3\sqrt{3}},\quad U_1=\frac{2}{\sqrt{3}},\quad L_1=-\frac{14}{3\sqrt{3}}\frac{e^2}{8a^3},$$

$$N_1=-\frac{10}{3\sqrt{3}}\frac{e^2}{8a^3},\quad M_1=\frac{2}{\sqrt{3}}\frac{e^2}{8a^3},\quad P_1=-\frac{8}{3\sqrt{3}}\frac{e^2}{8a^3},$$

$$S_2=S_1,\quad T_2=T_1,\quad U_2=-U_1,\quad L_2=L_1,\quad N_2=N_1,$$
$$M_2=-M_1,\quad P_2=P_1.$$

For the vibrations in the plane of the orbit, when $k=0$, the frequency equation is

$$\left(\sqrt{3}\frac{e^2}{a^3}-mq^2\right)(-mq^2)=4m^2\omega^2q^2;$$

the solution of this is

$$q=0 \text{ and } q=\left\{\sqrt{3}\frac{e^2}{ma^3}+4\omega^2\right\}^{\frac{1}{2}}=\left\{\frac{3\nu e^2}{mb^3}+\omega^2\right\}^{\frac{1}{2}}.$$

When $k=1$, the frequency equation is

$$\left(\frac{5}{4\sqrt{3}}\frac{e^2}{a^3}-mq^2\right)^2=\left(\frac{2}{\sqrt{3}}\frac{e^2}{8a^3}-2m\omega q\right)^2.$$

The solution of this equation is

$$q=\quad\omega\pm\sqrt{\frac{1}{\sqrt{3}}\frac{e^2}{ma^3}+\omega^2}=\omega\pm\sqrt{\frac{\nu e^2}{mb^3}},$$

$$q=-\omega\pm\sqrt{\frac{\sqrt{3}}{2}\frac{e^2}{ma^3}+\omega^2}=-\omega\pm\sqrt{\frac{3}{2}\frac{\nu e^2}{mb^3}-\tfrac{1}{2}\omega^2}.$$

When $k=2$ the frequencies are the same as when $k=1$; we have thus six frequencies corresponding to the six degrees of freedom of the three corpuscles in the plane of their undisturbed orbit.

For the vibration at right angles to the plane of this orbit, when $k=0$ the frequency equation is

$$\frac{\nu e^2}{b^3}-mq^2=0,$$

or

$$q=\sqrt{\frac{\nu e^2}{mb^3}}.$$

When $k=1$, the frequency equation is

$$\frac{\nu e^2}{b^3}-\frac{e^2}{\sqrt{3}a^3}-mq^2=0,$$

or $\qquad q=\pm\omega.$

In the case of three corpuscles, as in that of two, we see that when there is no rotation three of the periods are equal; these are separated when the corpuscles are in rotation.

Case of four corpuscles.

When $n=4$,

$$S_0=1+2\sqrt{2},\quad T_0=4\sqrt{2}+1,\quad U_0=0,\quad L_0=(6\sqrt{2}+2)\frac{e^2}{8a^3},$$

$$N_0=(6\sqrt{2}+1)\frac{e^2}{8a^3},\quad M_0=0,\quad P_0=(4\sqrt{2}+1)\frac{e^2}{8a^3},$$

$$S_1=-1,\qquad T_1=-1,\qquad U_1=2\sqrt{2},\qquad L_1=-2\frac{e^2}{8a^3},$$

$$N_1=-\frac{e^2}{8a^3},\quad M_1=2\sqrt{2}\frac{e^2}{8a^3},\quad P_1=-\frac{e^2}{8a^3},$$

$$S_2=-2\sqrt{2}+1,\quad T_2=-4\sqrt{2}+1,\quad U_2=0.\quad L_2=(-6\sqrt{2}+2)\frac{e^2}{8a^3},$$

$$N_2=(-6\sqrt{2}+1)\frac{e^2}{8a^3},\quad M_2=0,\quad P_2=(-4\sqrt{2}+1)\frac{e^2}{8a^3}.$$

When $k=0$, the frequency equation is

$$\left(\frac{3}{4}\frac{e^2}{a^3}(1+2\sqrt{2})-mq^2\right)(-mq^2)=4m^2\omega^2q^2;$$

the solution of which is

$$q=0,\quad q=\sqrt{\frac{3}{4}\frac{e^2}{a^3}(1+2\sqrt{2})+4\omega^2}=\sqrt{\frac{3\nu e^2}{mb^3}+\omega^2}.$$

When $k=1$, the frequency equation is

$$\left((6\sqrt{2}+2)\frac{e^2}{8a^3}-mq^2\right)^2=\left(2\sqrt{2}\frac{e^2}{8a^3}-2m\omega q\right)^2;$$

the solution of this is

$$q=\quad\omega\pm\sqrt{\frac{2\sqrt{2}+1}{4}\frac{e^2}{ma^3}+\omega^2}=\quad\omega\pm\sqrt{\frac{\nu e^2}{mb^3}},$$

$$q=-\omega\pm\sqrt{\frac{4\sqrt{2}+1}{4}\frac{e^2}{ma^3}+\omega^2}=-\omega\pm\sqrt{\frac{4\sqrt{2}+1}{2\sqrt{2}+1}\frac{\nu e^2}{mb^2}-\frac{2\sqrt{2}\omega^2}{2\sqrt{2}+1}}.$$

When $k=2$, the frequency equation is

$$\left(\frac{3}{4}\frac{e^2}{a^3}-mq^2\right)\left(\frac{3}{\sqrt{2}}\frac{e^2}{a^3}-mq^2\right)=4m^2\omega^2q^2.$$

Regarding this as a quadratic in q^2, we see that the roots are positive, so that the values of q are real and the arrangement is stable. The roots of the equation are

$$q^2=\frac{3}{8\sqrt{2}}(4+\sqrt{2})\frac{e^2}{ma^3}+2\omega^2$$
$$\pm\sqrt{\frac{9(4-\sqrt{2})^2}{128}\frac{e^4}{m^2a^6}+\frac{3(\sqrt{2}+4)}{2\sqrt{2}}\frac{e^2}{ma^3}\omega^2+4\omega^4}.$$

Let us now consider the motion at right angles to the plane of the orbit. When $k=0$, the frequency equation is

$$\frac{\nu e^2}{b^3}-mq^2=0,$$

or

$$q=\sqrt{\frac{\nu e^2}{mb^3}}.$$

When $k=1$, the frequency equation is

$$\frac{\nu e^2}{b^3}-(4\sqrt{2}+2)\frac{e^2}{8a^3}-mq^2=0,$$

or $\quad q=\pm\omega.$

When $k=2$, the frequency equation is

$$\frac{\nu e^2}{b^3}-\frac{8\sqrt{2}e^2}{8a^3}-mq^2=0,$$

or

$$q^2=\frac{8\sqrt{2}}{4\sqrt{2}+2}\omega^2-\frac{(4\sqrt{2}-2)}{4\sqrt{2}+2}\frac{\nu e^2}{mb^3}.$$

Thus, unless

$$\omega^2>\frac{4\sqrt{2}-2}{8\sqrt{2}}\frac{\nu e^2}{mb^3}>\cdot325\frac{\nu e^2}{mb^3}, \quad . \quad . \quad . \quad (1)$$

q^2 is negative, and the equilibrum is unstable, the four corpuscles then arranging themselves at the corner of a regular tetrahedron. When, however, ω is large enough to satisfy condition (1), four corpuscles will be in equilibrium when in steady motion in one plane at the corners of a square.

Case of five corpuscles.

When $n=5$, we have

$$S_0=5\cdot5056,\quad T_0=12\cdot1732,\quad U_0=0,\quad L_0=\frac{e^2}{8a^3}(17\cdot6788),$$

$$N_0=\frac{e}{8a^3}(18\cdot8408),\quad M_0=0,\quad P_0=\frac{e^2}{8a^3}(12\cdot1732),$$

$$S_1=-\cdot65,\quad T_1=1\cdot1609,\quad U_1=4\cdot856,\quad L_1=\frac{e^2}{8a^3}(\cdot511),$$

$$N_1=\frac{e^2}{8a^3}(2\cdot9716),\quad M_1=\frac{e^2}{8a^3}4\cdot856,\quad P_1=\frac{e^2}{8a^3}(1\cdot1609),$$

$$S_2=-2\cdot103,\ T_2=-7\cdot249,\ U_2=2\cdot103,\ L_2=-\frac{e^2}{8a^3}(9\cdot352),$$

$$N_2=-\frac{e^2}{8a^3}12\cdot4,\quad M_2=\frac{e^2}{8a^3}2\cdot103,\quad P_2=-\frac{e^2}{8a^3}7\cdot249.$$

The frequency equation when $k=0$ is

$$\left(\frac{3}{4}\,5\cdot5056\frac{e^2}{a^3}-mq^2\right)(-mq^2)=4m^2\omega^2q^2,$$

the solution of which is

$$q=0,\quad q=\sqrt{\frac{3\nu e}{b^3}+\omega^2}.$$

When $k=1$, the frequency equation is

$$\left(15\cdot87\frac{e^2}{8a^3}-mq^2\right)^2=\left(\frac{e^2}{8a^3}4\cdot856-2m\omega q\right)^2,$$

or

$$q=\omega\pm\sqrt{10{\cdot}918\frac{e^2}{8a^3}+\omega^2}=\omega\pm\sqrt{\frac{\nu e^2}{mb^3}},$$

$$q=-\omega\pm\sqrt{20{\cdot}726\frac{e^2}{8a^3}+\omega^2}.$$

When $k=2$,

$$\left(6\frac{e^2}{8a^3}-mq^2\right)\left(31{\cdot}24\frac{e^2}{8a^3}-mq^2\right)=\left(2{\cdot}103\frac{e^2}{8a^3}-2m\omega q\right)^2.$$

By applying the usual methods we find that all the roots of this equation are real, so that the steady motion of the five particles is stable for displacements in the plane of the orbit.

Let us now consider displacements at right angles to the plane of the orbit. When $k=0$ the frequency equation is

$$\frac{\nu e^2}{b^3}-mq^2=0,$$

the solution of which is

$$q=\sqrt{\frac{\nu e^2}{mb^3}}.$$

When $k=1$, the frequency equation is

$$m\omega^2-mq^2=0,$$

hence

$$q=\omega.$$

When $k=2$, the frequency equation is

$$\frac{\nu e^2}{b^3}-19{\cdot}42\frac{e^2}{8a^3}-mq^2=0,$$

or

$$\frac{19{\cdot}42}{11}m\omega^2+\frac{\nu e^2}{b^3}-\frac{19{\cdot}42}{11}\frac{\nu e^2}{b^3}-mq^2=0,$$

$$\frac{19{\cdot}42}{11}m\omega^2-\frac{8{\cdot}42}{11}\frac{\nu e^2}{b^3}-mq^2=0.$$

Hence, in order that the equilibrium may be stable,

$$\omega^2 \text{ must be } > \frac{8{\cdot}42}{19{\cdot}42}\frac{\nu e^2}{b^3} > {\cdot}433\frac{\nu e^2}{mb^3}.$$

Thus the five corpuscles are unstable when in one plane unless the angular velocity exceeds a certain value; the arrangement is stable, however, when the angular velocity is large.

Case of six corpuscles.

When $n=6$,

$$S_0=5+\frac{4}{\sqrt{3}},\quad T_0=17+\frac{16}{3\sqrt{3}},\quad U_0=0,\quad L_0=\left(22+\frac{28}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$N_0=\left(29+\frac{20}{3\sqrt{3}}\right)\frac{e^2}{8a^3},\quad M_0=0,\quad P_0=\left(17+\frac{16}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$S_1=1-\frac{2}{\sqrt{3}},\quad T_1=7-\frac{8}{3\sqrt{3}},\quad U_1=6+\frac{2}{\sqrt{3}},\quad L_1=\left(8-\frac{14}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$N_1=\left(13-\frac{10}{3\sqrt{3}}\right)\frac{e^2}{8a^3},\quad M_1=\left(7-\frac{8}{3\sqrt{3}}\right)\frac{e^2}{8a^3},\quad P_1=\left(7-\frac{8}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$S_2=-1-\frac{2}{\sqrt{3}},\quad T_2=-7-\frac{8}{3\sqrt{3}},\quad U_2=6-\frac{2}{\sqrt{3}},\quad L_2=\left(-8-\frac{14}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$N_2=\left(-13-\frac{10}{3\sqrt{3}}\right)\frac{e^2}{8a^3},\quad M_2=\left(6-\frac{2}{\sqrt{3}}\right)\frac{e^2}{8a^3},\quad P_2=\left(-7-\frac{8}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$S_3=-5+\frac{4}{\sqrt{3}},\quad T_3=-17+\frac{16}{3\sqrt{3}},\quad U_3=0,\quad L_3=\left(-22+\frac{28}{3\sqrt{3}}\right)\frac{e^2}{8a^3},$$

$$N_3=\left(-29+\frac{20}{3\sqrt{3}}\right)\frac{e^2}{8a^3},\quad M_3=0,\quad P_3=\left(-17+\frac{16}{3\sqrt{3}}\right)\frac{e^2}{8a^3}.$$

It is not necessary to write down all the frequency equations because, as we shall show, the arrangement of six corpuscles is unstable. For when $k=3$ the frequency equation is

$$\left(\frac{6}{8}\left(5+\frac{4}{\sqrt{3}}\right)\frac{e^2}{8a^3}-44\frac{e^2}{8a^3}-mq^2\right)\left(58\frac{e^2}{8a^3}-mq^2\right)=4m^2\omega^2q^2,$$

or
$$\left(-\frac{(14-8\sqrt{3})}{8}\frac{e^2}{a^3}-mq^2\right)\left(58\frac{e^2}{8a^3}-mq^2\right)=4m^2\omega^2q^2. \quad (1)$$

As $14-8\sqrt{3}$ is positive, we see that one of the roots of this equation for q^2 is negative, so that q is imaginary; this shows that the steady motion of 6 corpuscles in a ring is unstable, however rapid the rotation. We can, however, make the motion stable by putting a corpuscle at the centre; if we have a negative charge equal to that of p corpuscles at the centre of the ring the radial force it exerts on the sth corpuscle is $\frac{pe^2}{(a+\rho)^2}$, or $\frac{pe^2}{a^2}-\frac{2pe^2\rho}{a^3}$. Introducing this term into the expression for the radial force we find the frequency equation becomes

$$\left(\frac{3}{4}\frac{e^2}{a^3}S_0+\frac{3pe^2}{a^3}+L_k-L_0-mq^2\right)(N_0-N_k-mq^2)=(M_k-2m\omega q)^2.$$

Using this frequency equation, and supposing that $p=1$, *i. e.* that there is only one corpuscle at the centre of the hexagon, we get instead of (1),

$$\left(\frac{10+8\sqrt{3}}{8}\frac{e^2}{a^3}-mq^2\right)\left(\frac{58\,e^2}{8a^3}-mq^2\right)=4m^2\omega^2q^2. \quad . \quad (2)$$

The roots of this equation in q^2 are both positive, so that q is real and the equilibrium is stable.

Let us now investigate the conditions for stability for displacements at right angles to the plane of the orbit.

For the motion at right angles to the plane of the ring, the frequency equation when $k=3$ is

$$\frac{\nu e^2}{b^3}-\frac{pe^2}{a^3}-\frac{34\,e^2}{8a^3}-mq^2=0.$$

For this to represent the displacement of a stable system q^2 must be positive, so that if $p=1$

$$\frac{\nu e^2}{b^3}-\frac{e^2}{a^3}-\frac{34\,e^2}{8a^3}$$

must be positive ; we have, however,

$$\frac{\nu e^2}{b^3}=m\omega^2+\frac{e^2}{a^3}+\frac{e^2}{4a^3}\left(5+\frac{4}{\sqrt{3}}\right);$$

so that for

$$\frac{\nu e^2}{b^3}-\frac{e^2}{a^3}-\frac{34\,e^2}{8a^3}$$

to be positive

$$m\omega^2 \text{ must be greater than } \frac{12-4/\sqrt{3}}{21}\frac{\nu e^2}{b^3},\ i.\,e.\ \cdot 46\frac{\nu e^2}{b^3}.$$

Let us now consider the stability of the corpuscle at the centre of the ring: if it is displaced through a distance z at right angles to the ring, the equation of motion of the corpuscles is

$$m\frac{d^2z}{dt^2}=-\frac{\nu e^2}{b^3}z+\frac{6e^2}{a^3}z.$$

Thus if the motion is stable

$$\frac{\nu e^2}{b^3}>\frac{6e^2}{a^3},$$

or
$$m\omega^2 > \frac{15-4/\sqrt{3}}{24}\frac{\nu e^2}{b^3}, \textit{ i. e. } \cdot 53\frac{\nu e^2}{b^3}.$$

This value of ω^2 is greater than that required to make the equilibrium of the ring stable for displacements at right angles to its plane; if the central corpuscle, instead of being in the plane of the ring, was one side of the centre of the sphere of positive electrification while the ring was on the other side, the rotation required to make the equilibrium of the detached corpuscle stable would be less than when it was in the plane of the ring; for equilibrium the distance of the detached corpuscle from the centre of the sphere must be six times the distance of the plane of the ring from that point.

Conditions for the stability of rings containing more than six corpuscles.

I find that a single corpuscle in the centre is sufficient to make rings of 7 and 8 corpuscles stable; in the latter case, however, one of the values of q^2 though positive is exceedingly small. When the number of corpuscles exceeds 8 the number of central corpuscles required to ensure stability increases very rapidly with the number of corpuscles in the ring.

The frequency equation is

$$\left(\frac{3}{4}\frac{e^2 S_0}{a^3}+\frac{3pe^2}{a^3}-(L_0-L_k)-mq^2\right)\left(N_0-N_k-mq^2\right)=(M_k-2m\omega q)^2.$$

Now N_0-N_k is always positive and M is small compared with L and N; hence this equation will have real roots if

$$\frac{3}{4}\frac{e^2 S_0}{a^3}+\frac{3pe^2}{a^3}-(L_0-L_k)$$

is positive. The greatest value of L_0-L_k is got by putting $k=n/2$ when n is even, and $=(n-1)/2$ when n is odd: hence the condition that the values of q should be real, *i. e.* that the equilibrium of the ring should be stable, is

$$\frac{3pe^2}{a^3} > (L_0-L_{\frac{n}{2}})-\frac{3}{4}\frac{e^2 S_0}{a^3} \text{ when } n \text{ is even,}$$

and

$$\frac{3pe^2}{a^3} > (L_0-L_{\frac{n-1}{2}})-\frac{3}{4}\frac{e^2 S_0}{a^3} \text{ when } n \text{ is odd.}$$

From this equation we can calculate the least value of p which will make a ring of n corpuscles stable. The values of

p for a series of values of n are given in the following table:—

n ...	5	6	7	8	9	10	15	20	30	40
p ...	0	1	1	1	2	3	15	39	101	232

For large values of n the values of p are proportional to n^3. When p is greater than one, the internal corpuscles necessary to produce equilibrium cannot all be at the centre of the sphere, they will separate until their repulsions are balanced by the attraction of the positive electricity in the sphere. Thus when there are two internal corpuscles, as when $n=9$, these two will separate and will form a pair with the line joining them parallel to the plane of the ring. If we assume, as is approximately the case, that the pair of equal corpuscles exerts at external points the same force as a double charge placed at a point midway between them, the preceding theory will apply, and the system consisting of the ring of 9 and the pair of corpuscles will be in stable equilibrium. When $n=10$, the internal corpuscles must be three in number; these three will arrange themselves at the corners of an equilateral triangle, and the system of 13 corpuscles will consist of a ring of 10 and a triangle of 3, the planes of the ring and triangle being parallel but not coincident; the corpuscles are all supposed to be in rapid rotation round the diameter of the sphere drawn at right angles to the planes of the ring. For a ring of 12 corpuscles we require 7 inside, but 7 corpuscles, as we have seen, cannot form a single ring, but will arrange themselves as a ring of 6 with one at the centre. Thus the system of 19 corpuscles will consist of an outer ring of 12, an inner ring of 6 in a plane parallel to the outer ring, and one corpuscle along the axis of rotation.

In this way we see that when we have a large number of corpuscles in rapid rotation they will arrange themselves as follows:—The corpuscles form a series of rings, the corpuscles in one ring being approximately in a plane at right angles to the axis of rotation, the number of particles in the rings diminishing as the radius of the ring diminishes. If the corpuscles can move at right angles to the plane of their orbit, the rings will be in different planes adjusting themselves so that the repulsion between the rings is balanced by the attraction exerted by the positive electrification of the sphere in which they are placed. We have thus in the first place a sphere of uniform positive electrification, and inside this sphere a number of corpuscles arranged in a series of parallel rings, the number of corpuscles in a ring varying from ring to ring: each corpuscle is travelling at a high speed round the circumference of the ring in which it is situated, and the rings are so arranged that those which contain a large number of corpuscles are near the surface of the sphere, while those in which there are a smaller number of corpuscles are more in the inside.

If the corpuscles, like the poles of the little magnets in Mayer's experiments with the floating magnets, are constrained to move in one plane, they would, even if not in rotation, be in equilibrium when arranged in the series of rings just described. The rotation is required to make the arrangement stable when the corpuscles can move at right angles to the plane of the ring.

Application of the preceding Results to the Theory of the Structure of the Atom.

We suppose that the atom consists of a number of corpuscles moving about in a sphere of uniform positive electrification: the problems we have to solve are (1) what would be the structure of such an atom, *i. e.* how would the corpuscles arrange themselves in the sphere; and (2) what properties would this structure confer upon the atom. The solution of (1) when the corpuscles are constrained to move in one plane is indicated by the results we have just obtained—the corpuscles will arrange themselves in a series of concentric rings. This arrangement is necessitated by the fact that a large number of corpuscles cannot be in stable equilibrium when arranged as a single ring, while this ring can be made stable by placing inside it an appropriate number of corpuscles. When the corpuscles are not constrained to one plane, but can move about in all directions, they will arrange themselves in a series of concentric shells; for we can easily see that, as in the case of the ring, a number of corpuscles distributed over the surface of a shell will not be in stable equilibrium if the number of corpuscles is large, unless there are other corpuscles inside the shell, while the equilibrium can be made stable by introducing within the shell an appropriate number of other corpuscles.

The analytical and geometrical difficulties of the problem of the distribution of the corpuscles when they are arranged in shells are much greater than when they are arranged in rings, and I have not as yet succeeded in getting a general solution. We can see, however, that the same kind of properties will be associated with the shells as with the rings; and as our solution of the latter case enables us to give definite results, I shall confine myself to this case, and endeavour to show that the properties conferred on the atom by this ring structure are analogous in many respects to those possessed by the atoms of the chemical elements, and that in particular the properties of the atom will depend upon its atomic weight in a way very analogous to that expressed by the periodic law.

Let us suppose, then, that we have N corpuscles each carrying a charge e of negative electricity, placed in a sphere of positive electrification, the whole charge in the sphere being equal to Ne; let us find the distribution of the corpuscles when they are arranged in what we may consider to be the simplest way, *i. e.* when the number of rings is a minimum, so that in each ring there are as nearly as possible as many corpuscles as it is possible for the corpuscles inside to hold in equilibrium. Let us suppose that the number of internal corpuscles required to make the equilibrium of a ring of n corpuscles stable is $f(n)$. The value of $f(n)$ for a series of values of n is given in the table on page 254; in that table $f(n)$ is denoted by p. The number of corpuscles in the outer ring n_1 will then be determined by the condition that $N-n_1$, the number of corpuscles inside, must be just sufficient to keep the ring of n_1 corpuscles in equilibrium, *i. e.*, n_1 will be determined by the equation

$$N-n_1=f(n_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If the value of n_1 got from this equation is not an integer we must take the integral part of the value.

To get n_2, the number of corpuscles in the second ring, we notice that there must be $N-n_1-n_2$ corpuscles inside; hence n_2 is given by the equation

$$N-n_1-n_2=f(n_2).$$

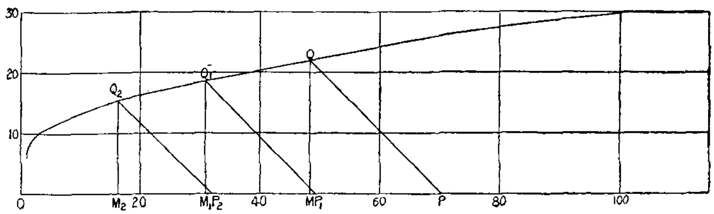
Similarly, n_3, n_4, ..., the number of corpuscles in the 3rd, 4th, &c. rings reckoned from the outside, are given by

$$N-n_1-n_2-n_3=f(n_3),$$
$$N-n_1-n_2-n_3-n_4=f(n_4).$$

These equations can be solved very rapidly by a graphical method. Draw the graph whose abscissa $=f(n)$ and whose ordinate is n. The values of $f(n)$ for a series of values of n are given on page 254; from these values the curve fig. 1 has been constructed.

To find how a number of corpuscles equal to N will arrange themselves, measure off on the axis of abscissæ a distance from O equal to N. Let OP be this distance, through P draw PQ inclined at an angle of 135° to the horizontal axis, cutting the curve in Q, draw the ordinate QM; then the integral part of QM will be the value of n_1, the number of

Fig. 1.



corpuscles in the first ring reckoned from the outside. For evidently

$$OM = f(QM),$$

and OM=ON−NM, and since PQ is inclined at 45° to the axis, NM=OM; hence

$$ON - QM = f(QM).$$

Comparing this with equation (1) we see that the integral part of QM is the value of n_1.

To get the value of n_2, the number of corpuscles in the second ring, we mark off the abscissa $OP_1 = N - n_1$ (if QM is an integer P_1 will coincide with M), then from P_1 draw P_1Q_1 parallel to PQ cutting the curve in Q_1; the integral part of Q_1M_1 will be the value of n_2. To get n_3 mark off the abscissa $OP_2 = N - n_1 - n_2$, and draw P_2Q_2 parallel to PQ; the integral part of Q_2M_2 will be the value of n_3. In this way we can in a very short time find the configuration.

The following table, which gives the way in which various numbers of corpuscles group themselves, has been calculated in this way; the numbers range downwards from 60 at intervals of 5.

Number of corpuscles	60.	55.	50.	45.	40.	35.
Number in successive rings...	20 16 13 8 3	19 16 12 7 1	18 15 11 5 1	17 14 10 4	16 13 8 3	16 12 6 1
Number of corpuscles	30.	25.	20.	15.	10.	5.
Number in successive rings...	15 10 5	13 9 3	12 7 1	10 5	8 2	5

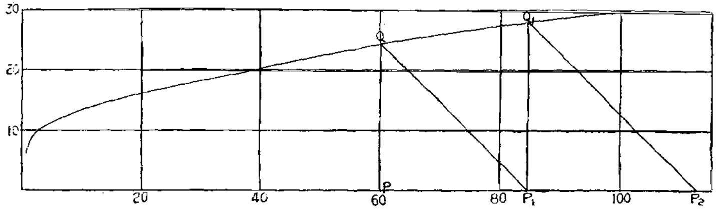
We give also the entire series of arrangement of corpuscles for which the outer ring consists of 20 corpuscles.

Number of corpuscles	59.	60.	61.	62.	63.	64.	65.	66.	67.
Number in successive rings...	20 16 13 8 2	20 16 13 8 3	20 16 13 9 3	20 17 13 9 3	20 17 13 10 3	20 17 13 10 4	20 17 14 10 4	20 17 14 10 5	20 17 15 10 5

59 is the smallest number of corpuscles which can have an outer ring of 20, while when the number of corpuscles is greater than 67 the outer ring will contain more than 20 corpuscles.

Let us now consider the connexion between these results and the properties possessed by the atoms of the chemical elements. We suppose that the mass of an atom is the sum of the masses of the corpuscles it contains, so that the atomic weight of an element is measured by the number of corpuscles in its atom. An inspection of the results just given will show that systems built up of rings of corpuscles in the way we have described, will possess properties analogous to some of those possessed by the atom. In the first place, we see that the various arrangements of the corpuscles can be classified in families, the grouping of the corpuscles in the various members of the family having certain features in common. Thus, for example, we see that the group of 60 corpuscles consists of the same rings of corpuscles as the group of 40 with an additional ring of 20 corpuscles round it, while the group of 40 consists of the same series of rings as the group of 24 with an additional ring outside, while 24 is the group 11 with an additional ring. To continue the series for larger numbers of corpuscles, take the curve $x=f(y)$ when $f(n)$ is the number of corpuscles that must be placed

Fig. 2.



inside a ring of n corpuscles to make it stable. Let Q be the point on this curve corresponding to 60 corpuscles, *i. e.* OP=60, from Q draw QP_1 inclined at an angle of 135° to the axis of x; then the number of corpuscles represented by OP_1 will be arranged like the 60 corpuscles with an addition ring of Q_1P_1 corpuscles (fig. 2). To find the next member of the family, draw Q_1P_2 parallel to QP_1 cutting the axis of x in P_2, then OP_2 will represent the number of corpuscles in the next member of the family; and by continuing the process we can find the successive members. Thus we see that we can divide the various groups of atoms into series such that each member of the series is derived from the preceding member (*i. e.* the member next below it in atomic weight) by adding to it another ring of corpuscles. We should expect the atoms formed by a series of corpuscles of this kind to have many points of resemblance. Take, for example, the vibrations of the corpuscles; these may be divided into two sets:—(1) Those arising from the rotation of the corpuscles around their orbits: if all the corpuscles in one atom have the same angular velocity, the frequency of the vibrations produced by the rotation of the ring of corpuscles is proportional to the number of corpuscles in the ring; and thus in the spectrum of each element in the series there would be a series of frequencies bearing the same ratio to each other, the ratio of the frequencies being the ratios of the numbers in the various rings.

The second system of vibrations are those arising from the displacement of the ring from its circular figure. If now the distance of a corpuscle in the outer ring from a corpuscle in the collection of rings inside it is great compared with the distance of the second corpuscle from its nearest neighbour on its own ring, the effect of the outer ring of corpuscles on the inner set of rings will only "disturb" the vibrations of the latter without fundamentally altering the character of their vibrations. Thus for these vibrations, as well as for those due to the rotations, the sequence of frequencies would present much the same features for the various elements in the series; there would be in the spectrum corresponding groups of associated lines. We regard a series of atoms formed in this way, *i. e.* when the atom of the pth member is formed from that of the $(p-1)$th by the addition of a single ring of corpuscles, as belonging to elements in the same group in the arrangement of the elements according to the periodic law; *i. e.*, they form a series which, if arranged according to Mendeléef's table, would all be in the same vertical column.

The gradual change in the properties of the elements which takes place as we travel along one of the horizontal rows in Mendeléef's arrangement of the elements, is also illustrated by the properties possessed by these groups of corpuscles. Thus consider the series of arrangements of the corpuscles given on p. 258, in all of which the outer ring contains 20 corpuscles An outer row of 20 corpuscles first occurs with 59 corpuscles; in this case the number of corpuscles inside is only just sufficient to make the outer ring stable; this ring will therefore be on the verge of instability, and when the corpuscles in this ring are displaced the forces of restitution urging them back to their original position will be small. Thus when this ring is subjected to disturbances from an external source, one or more corpuscles may easily be detached from it; such an atom therefore will easily lose a negatively electrified corpuscle, and thus acquire a charge of positive electricity; such an atom would behave like the atom of a strongly electropositive element. When we pass from 59 to 60 corpuscles the outer ring is more stable, because there is an additional corpuscle inside it; the corresponding atom will thus not be so electropositive as that containing only 59 corpuscles. The addition of each successive corpuscle will make it more difficult to detach corpuscles from the outer ring, and will therefore make the atom less electropositive. When the stability of the outer ring gets very great, it may be possible for one or more corpuscles to be on the surface of the atom without breaking up the ring; in this case the atom could receive a charge of negative electricity, and would behave like the atom of an electronegative element. The increase in the stability of the ring, and consequently in the electronegative character of the atom, would go on increasing until we had as many as 67 corpuscles, when the stability of the outer ring would be at a maximum. A great change in the properties of the atom would occur with 68 corpuscles, for now the number of corpuscles in the outer ring increases to 21; these 21 corpuscles are, however, only just stable, and would, like the outer ring of 20 in the arrangement of the 59 corpuscles, readily lose a corpuscle and so make the atom strongly electropositive.

The properties of the groups of 59 and 67 corpuscles, which are respectively at the beginning and end of the series which has an outer ring of 20 corpuscles, deserve especial consideration. The arrangement of corpuscles in the group of 59, although very near the verge of instability, and therefore very liable to lose a corpuscle and thereby acquire a positive charge, would not be able to retain this charge. For when it had lost a corpuscle, the 58 corpuscles left would arrange themselves in the grouping corresponding to 58 corpuscles which is the last to have an outer ring of 19 corpuscles; this ring is therefore exceedingly stable so that no further corpuscles would escape from it, while the positive charge on the system due to the escape of the 59th corpuscle would attract the surrounding corpuscles. Thus this arrangement could not remain permanently charged; for as soon as one corpuscle had escaped it would be replaced by another. An atom constituted in this way would be neither electropositive nor electronegative, but one incapable of receiving permanently a charge of electricity.

The group containing 60 corpuscles would be the most electropositive of the series; but this could only lose one corpuscle; *i. e.* acquire a charge of one unit of positive electricity; for if it lost two we should have 58 corpuscles—as when the group of 59 had lost one corpuscle—and in this case the system would be even more likely than the other to attract external corpuscles, for it would have a charge of two units of positive electricity instead of one. Thus the system containing 60 corpuscles would get charged with one, but only one, unit of positive electricity: it would therefore act like the atom of a monovalent electropositive element.

The group containing 61 corpuscles would not part with its corpuscles so readily as the group of 60, but on the other hand it could afford to lose two, as it is not until it has lost three that its corpuscles are reduced to 58, when, as we have seen, it begins to acquire fresh corpuscles. Thus this system might get charged with two units of positive electricity, and would act like the atom of a divalent electropositive element. Similarly the group of 62, though less liable even than the 61 to lose its corpuscles, could, on the other hand, lose 3 without beginning to recover its corpuscles; it could thus acquire a charge of 3 units of positive electricity, and would act like the atom of a trivalent electropositive element.

Let us now go to the groups at the other end of the series and consider the properties of the last of the series, the group of 67 corpuscles. The outer ring would be very stable, but if the system acquired another corpuscle, the 68 corpuscles would arrange themselves with a ring of 21 corpuscles on the outside; as 68 is the smallest number of corpuscles with an outer ring of 21, the ring is very nearly unstable and easily loses a corpuscle. Thus the group of 67 corpuscles, as soon as it acquires a negative charge, would lose it again, and the system, like the group of 59, would be incapable of being permanently charged with electricity—it would act like the atom of an element of no valency.

The group of 66 would be the most electronegative of the series, but this would only be able to retain a charge of one unit of negative electricity; for if it acquired 2 units there would be 68 corpuscles, an arrangement which, as we have seen, rapidly loses its corpuscles. This group of 66 would therefore act like the atom of a monovalent electronegative element.

The group of 65 would be less liable than that of 66 to acquire negative corpuscles, but, on the other hand, it would under suitable circumstances be able to retain 2 corpuscles, and thus be charged with 2 units of negative electricity, and would act like the atom of a divalent electronegative element.

Similarly, the group of 64 would act like the atom of a trivalent electronegative element, and so on.

Thus, if we consider the series of arrangements of corpuscles having on the outside a ring containing a constant number of corpuscles, we have at the beginning and end systems which behave like the atoms of an element whose atoms are incapable of retaining a charge of either positive or negative electricity; then (proceeding in the order of increasing number of corpuscles) we have first a system which behaves like the atom of a monovalent electropositive element, next one which behaves like the atom of a divalent electropositive element, while at the other end of the series we have a system which behaves like an atom with no valency, immediately preceding this, one which behaves like the atom of a monovalent electronegative element, while this again is preceded by one behaving like the atom of a divalent electronegative element.

This sequence of properties is very like that observed in the case of the atoms of the elements.

Thus we have the series of elements :

He	Li	Be	B	C	N	O	F	Ne.
Ne	Na	Mg	Al	Si	P	S	Cl	Arg.

The first and last element in each of these series has no valency, the second is a monovalent electropositive element, the last but one is a monovalent electronegative element, the third is a divalent electropositive element, the last but two a divalent electronegative element, and so on.

When atoms like the electronegative ones, in which the corpuscles are very stable, are mixed with atoms like the electropositive ones, in which the corpuscles are not nearly so firmly held, the forces to which the corpuscles are subject by the action of the atoms upon each other may result in the detachment of corpuscles from the electropositive atoms and their transference to the electronegative. The electronegative atoms will thus get a charge of negative electricity, the electropositive atoms one of positive, the oppositely charged atoms will attract each other, and a chemical compound of the electropositive and electronegative atoms will be formed.

Just as an uncharged conducting sphere will by electrostatic induction attract a corpuscle in its neighbourhood, so a corpuscle outside an atom will be attracted, even though the atom has not become positively charged by losing a corpuscle. When the outside corpuscle is dragged into the atom there will be a diminution in the potential energy, the amount of this diminution depending on the number of corpuscles in the atom. If now we have an atom A such that loss of potential energy due to the fall into the atom of a corpuscle from outside is greater than the work required to drag a corpuscle from an atom B of a different kind, then an intimate mixture of A and B atoms will result in the A atoms dragging corpuscles from the B atoms, thus the A atoms will get negatively, the B atoms positively electrified, and the oppositely electrified atoms will combine, forming a compound such as $A_{-}B_{+}$; in such a case as this chemical combination might be expected whenever the atoms were brought into contact. Even when the loss of potential energy when a corpuscle falls into A is less than the work required to drag a corpuscle right away from B, the existence of a suitable physical environment may lead to chemical combination between A and B. For when a corpuscle is dragged out of and away from an atom a considerable portion of the work is spent on the corpuscle after it has left the atom, while of the work gained when a corpuscle falls into an atom, the proportion done outside to that done inside the atom is smaller than the proportion for the corresponding quantities when the corpuscle is dragged out of an atom. Thus, though the work required to move a corpuscle from B to an infinite distance may be greater than that gained when a corpuscle moves from an infinite distance into A, yet the work gained when a corpuscle went from the surface of A into its interior might be greater than the work required to move a corpuscle from the interior to the surface of B. In this case anything which diminished the forces on the corpuscle when they got outside the atom, as, for example, the presence of a medium of great specific inductive capacity such as water, or contact with a metal such as platinum black, would greatly increase the chance of chemical combination.

The Existence of Secondary Groups of Corpuscles within the Atom.

The expression given on p. 238 for the radius of a ring of corpuscles shows that it depends on $\nu e/b^3$, where νe is the amount of positive electrification within a sphere of radius b: thus ve/b^3 is equal to $\frac{4\pi}{3}\rho$, where ρ is the density of the positive electrification in the sphere: thus, if the density of the electrification be kept constant, the radius of the ring will be independent of the size of the sphere. Now let us take a large sphere and place within it a ring of such a size that the ring would be in stable equilibrium if its centre were at the centre of the sphere. To fix our ideas, let us take the case of three corpuscles at the corners of an equilateral triangle, and place this triangle so that its centre O′ is no longer at the centre of the sphere: we can easily see that the corpuscles will remain at the corners of an equilateral triangle of the same size, and that the triangle will move like a rigid body acted upon by a force proportional to the distance of its centre from O the centre of the sphere. To prove this we notice that the repulsion between the corpuscles is the same as when the centre of the triangle is at O. The attraction of the sphere on a corpuscle P is proportional to OP, and so may be resolved into two forces, one proportional to O′P along PO′ (O′ is the centre of the triangle) and the other proportional to OO′ acting along O′O. Now the corpuscles are by hypothesis in equilibrium under their mutual repulsions, and the attraction to the centre proportional to O′P: thus the relative position of the corpuscles will remain unaltered, and the system of three corpuscles will move as a rigid body under a central force acting on its centre of gravity proportional to the distance of that point from the centre of the sphere.

The three corpuscles will, at a point whose distance from their centre is large compared with a side of the triangle, produce the same effect as if the charges on the three corpuscles were condensed at the centre of the triangle; they will thus at such points act like a unit, and the results we have previously obtained for single corpuscles may be extended to the case when the single corpuscles are replaced by rings of corpuscles which would by themselves be in equilibrium. It should be noted that the atom in which these systems are placed must be large enough to allow these rings of corpuscles—sub-atoms we may call them, to be separated by distances considerably greater than the distance between the corpuscles in one of the rings.

If we regard the atoms of the heavier elements as produced by the coalescence of lighter atoms, it is reasonable to suppose that the corpuscles in the heavier atoms may be arranged in secondary groups or sub-atoms, each of these groups acting as a unit. When the corpuscles are done up in bundles in this way, it is possible to have stability when these bundles are arranged in a ring with a smaller number of corpuscles inside than when the corpuscles in the bundles are arranged at equal intervals round the circumference of the ring. Thus, take the case of a ring of 30 corpuscles; if these were arranged at equal intervals, 101 corpuscles would be required inside the ring to make it stable. If, however, the 30 corpuscles were grouped in ten sets of three each, only $3 \times 3 = 9$ corpuscles in the interior would be required to make the arrangement stable.

Constitution of the Atom of a Radioactive Element.

Our study of the stability of systems of corpuscles has made us acquainted with systems which are stable when the corpuscles are rotating with an angular velocity greater than a certain value, but which become unstable when the velocity falls below this value. Thus, to take an instance, we saw (p. 249) that four corpuscles can be stable in one plane at the corners of a square, if they are rotating with an angular velocity greater than $\cdot 325 ve^2/mb^3$, but become unstable if the velocity falls below this velocity, the corpuscles in this case tending to place themselves at the corners of a tetrahedron. Consider now the properties of an atom containing a system of corpuscles of this kind, suppose the corpuscles were originally moving with velocities far exceeding the critical velocity; in consequence of the radiation from the moving corpuscles, their velocities will slowly—very slowly—diminish; when, after a long interval, the velocity reaches the critical velocity, there will be what is equivalent to an explosion of the corpuscles, the corpuscles will move far away from their original positions, their potential energy will decrease, while their kinetic energy will increase. The kinetic energy gained in this way might be sufficient to carry the system out of the atom, and we should have, as in the case of radium, a part of the atom shot off. In consequence of the very slow dissipation of energy by radiation the life of the atom would be very long. We have taken the case of the four corpuscles as the type of a system which, like a top, requires for its stability a certain amount of rotation. Any system possessing this property would, in consequence of the gradual dissipation of energy by radiation, give to the atom containing it radioactive properties similar to those conferred by the four corpuscles.